



5.9 Angle between vectors

Douglas Wilhelm Harder, LEL, M.Math.

dwharder@uwaterloo.ca

dwharder@gmail.com





Introduction

- In this topic, we will
 - Define the angle between two vectors
 - Look at some examples
 - Consider an application





Angle between vectors

- In a previous lecture introducing vectors, we deduced from the definition of the trigonometric functions that

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2}$$

- This was shown for 2-dimensional real vectors





Angle between vectors

- We have seen the Cauchy-Bunyakovsky-Schwarz inequality for all inner products and their induced 2-norms:

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$$

- Therefore, $-\|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \leq \langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$

- Thus, for non-zero $\mathbf{u}$ and $\mathbf{v}$ we have that $-1 \leq \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} \leq 1$

- Therefore, for an arbitrary inner-product space,
we will define

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2}$$





Angle between vectors

- Now, the arc cosine function returns a value in the interval

$$[0, \pi]$$

- If $\theta = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are positive scalar multiples of each other
- If $0 < \theta < \pi/2$, then $\mathbf{u}$ and $\mathbf{v}$ “point” in the same general direction
- If $\theta = \pi/2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal
- If $\pi/2 < \theta < \pi$, then $\mathbf{u}$ and $\mathbf{v}$ “point” in opposite directions
- If $\theta = \pi$, then $\mathbf{u}$ and $\mathbf{v}$ are negative scalar multiples of each other





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3 \cdot 12 + 4 \cdot 5 = 56$$

$$\|\mathbf{u}\|_2 = \sqrt{3^2 + 4^2} = 5$$

$$\|\mathbf{v}\|_2 = \sqrt{12^2 + 5^2} = 13$$

$$\cos(\theta) = \frac{56}{5 \cdot 13}$$

- Therefore, $\theta \approx 0.5325 \approx 30.51^\circ$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ -6 \end{pmatrix}$
 $\langle \mathbf{u}, \mathbf{v} \rangle = 2 \cdot (-3) + 1 \cdot 2 + 2 \cdot (-6) = -16$

$$\|\mathbf{u}\|_2 = \sqrt{2^2 + 1^2 + 2^2} = 3$$

$$\|\mathbf{v}\|_2 = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$\cos(\theta) = \frac{-16}{3 \cdot 7}$$

- Therefore, $\theta \approx 2.4370 \approx 139.63^\circ$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 6 \\ -9 \end{pmatrix}$

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1 \cdot (-3) + (-2) \cdot 6 + 3 \cdot (-9) = -42$$

$$\|\mathbf{u}\|_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\|_2 = \sqrt{3^2 + 6^2 + 9^2} = \sqrt{126}$$

$$\cos(\theta) = \frac{-42}{42}$$

– Therefore, $\theta = \pi = 180^\circ$





Examples

- Consider $\mathbf{u} = \begin{pmatrix} 0.3 \\ -0.5 \\ 0.1 \\ 1.2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0.2 \\ -2.4 \\ -0.6 \\ -1 \end{pmatrix}$

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= 0.3 \cdot 0.2 + (-0.5) \cdot (-2.4) + 0.1 \cdot (-0.6) + 1.2 \cdot (-1) \\ &= 0.06 + 1.2 - 0.06 - 1.2 = 0\end{aligned}$$

$$\|\mathbf{u}\|_2 = \sqrt{0.3^2 + 0.5^2 + 0.1^2 + 1.2^2} = \sqrt{1.79}$$

$$\|\mathbf{v}\|_2 = \sqrt{0.2^2 + 2.4^2 + 0.6^2 + 1^2} = \sqrt{7.16}$$

$$\cos(\theta) = \frac{0}{3.58}$$

– Therefore, $\theta = \frac{\pi}{2} = 90^\circ$





Application

- If two points are known to be on the surface of a circle, sphere, or n -dimensional ball of radius r ,
and we calculate the angle q between those points
that angle times r is the distance between those points

$$\mathbf{u}_{\text{Ottawa}} = 6371 \begin{pmatrix} \cos(-75^\circ 41') \cos(45^\circ 25') \\ \sin(-75^\circ 41') \cos(45^\circ 25') \\ \sin(45^\circ 25') \end{pmatrix} \approx \begin{pmatrix} 1106 \\ -4333 \\ 4538 \end{pmatrix}$$

$$\mathbf{u}_{\text{Mecca}} = 6371 \begin{pmatrix} \cos(39^\circ 49') \cos(21^\circ 25') \\ \sin(39^\circ 49') \cos(21^\circ 25') \\ \sin(21^\circ 25') \end{pmatrix} \approx \begin{pmatrix} 4556 \\ 3797 \\ 2326 \end{pmatrix}$$





Application

- Calculating the cosine of the angle, we have:

$$\frac{\langle \mathbf{u}_{\text{Ottawa}}, \mathbf{u}_{\text{Mecca}} \rangle}{6371^2} \approx \frac{1106 \cdot 4556 - 4333 \cdot 3797 + 4538 \cdot 2326}{6371^2}$$
$$= \frac{-858077}{40589641} \approx -0.02114$$

- Therefore, $\theta \approx 1.5919$ and the distance is
 $6371 \cdot 1.5919 = 10142$





Summary

- Following this topic, you now
 - Know the definition of the angle between two vectors
 - Have seen a few examples
 - Have considered an application calculating distance





References

- [1] https://en.wikipedia.org/wiki/Cauchy-Bunyakovsky-Schwarz_inequality
- [2] https://en.wikipedia.org/wiki/Inner_product_space





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

<https://www.rbg.ca/>

for more information.





Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.

